

# Spectra of sparse non-Hermitian random matrices: an analytical solution

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We present the exact analytical expression for the spectrum of a sparse non-Hermitian random matrix ensemble, generalizing two standard results in random-matrix theory: this analytical expression constitutes a non-Hermitian version of the Kesten-McKay measure as well as a sparse realization of Girko's elliptic law. Our exact result opens new perspectives in the study of several physical problems modelled on sparse random graphs which are locally tree like. In this context, we show analytically that the convergence rate of a transport process on a very sparse graph depends in a nonmonotonic way upon the degree of symmetry of the graph edges.

PACS numbers: 02.50.-r, 02.10.Yn, 89.75.Hc

*Introduction* Random matrices are an indispensable tool in the study of various problems throughout physics and mathematics. They pervade diverse areas as nuclear physics [1], number theory [2], quantum chaos and disordered mesoscopic systems [3], high-dimensional statistics [4] and information theory [5], etc. Owing to the ubiquitousness of random-matrix inspired models, exact analytical results lead to further insights in several fields.

The lions' share of the research on random-matrix theory has been dedicated either to dense ensembles or to sparse Hermitian ensembles (the latter in the context of spectral graph theory). From the abundance of results known for dense ensembles [6–8], we mention here the celebrated *Wigner's law* [9] and *Girko's elliptic law* [10] for, respectively, Hermitian and non-Hermitian matrices. A large number of results for spectra of graphs have also been derived [11]. Probably, the most simple of these results is the *Kesten-McKay* measure for the spectra of undirected regular graphs [12]. Exact analytical expressions for spectra of graphs are important for processes on graphs: the computation of effective resistances [13], synchronization in the presence of noise [14], mixing times of transport processes [15], bounds on learning in information theory [16], models for quantum chaos [17], combinatorial problems on graphs [18], thermodynamics of crystalline lattices [19], etc.

While many exact analytical results for sparse Hermitian ensembles are known, exact expressions for the spectra of their non-Hermitian counterparts have not yet been derived. This might be due to the difficulty of applying the standard manipulations from Hermitian random matrix theory to the non-Hermitian case [7, 8]. We refer to [20, 21] as one of the few works which have developed exact results on sparse non-Hermitian matrices. In [20, 21] Girko's law has been derived for sparse ensembles at high connectivities. Despite these efforts, analytical results are sparse and at this moment even an equivalent of the paradigmatic Kesten-McKay law is not known for non-Hermitian ensembles. A non-Hermitian version of the

Kesten-McKay measure would be an important tool in the development of an exact description of certain physical processes taking place on graphs with oriented edges. This is especially true if this measure would interpolate continuously between a fully undirected and a fully directed graph. As examples of areas which could benefit from such a result, we mention biased diffusion processes [22], synchronization [23], neural networks [24, 25] and non-Hermitian quantum systems, such as tight-binding models with imaginary vector potentials [26, 27] and quantum dissipative systems [28].

In this work we present an exact analytical formula for the spectrum of a sparse non-Hermitian random matrix ensemble, opening perspectives in several fields which benefit from random matrix theory. Such an exact result is possible thanks to recent advances in the theory of sparse random matrices with a local-tree like structure [21, 31, 32]. Our expression reduces to Girko's elliptic law in its highly connected limit and to the Kesten-McKay law in its Hermitian limit. Hence, it is a non-Hermitian equivalent of the Kesten-McKay law and a sparse realization of Girko's elliptic law. Our result is remarkably simple and allows to study the evolution of the spectrum as a function of the degree of symmetry in the graph edges. We illustrate the interest of this non-Hermitian Kesten-McKay law through one physical application: the exact calculation of the convergence rate of a stochastic diffusion on a partially-oriented random graph. This process can be seen as a toy model for vehicular traffic as well as biological transport [22].

*The resolvent equations* The complex eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$  of a non-Hermitian random matrix  $\mathbf{A}_N$  drawn from an ensemble are defined as the roots of the polynomial  $p(\lambda) = \det[\mathbf{A}_N - \lambda \mathbf{I}]$  of degree  $N$  in  $\lambda$ . We define the spectrum of the ensemble of random matrices  $\mathbf{A}_N$  at a certain point  $\lambda = x + iy$  by

$$\rho(\lambda) \equiv \lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^N \delta_{\lambda_i}(\mathbf{A}_N), \quad (1)$$

assuming self-averaging of this quantity for  $N \rightarrow \infty$ . The spectrum can be formally related to the resolvent  $\mathbf{G}_{\mathbf{A}}(\lambda)$  of  $\mathbf{A}$ , defined by  $\mathbf{G}_{\mathbf{A}}(\lambda) \equiv (\lambda - \mathbf{A})^{-1}$ , through the equation  $\rho(\lambda) = \lim_{N \rightarrow \infty} (N\pi)^{-1} \partial^* \text{Tr} \mathbf{G}_{\mathbf{A}}(\lambda)$ , where  $\partial^* = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$  and  $(\dots)^*$  denotes complex conjugation. The above equation follows from the identity  $\partial^* \lambda^{-1} = \pi \delta(x) \delta(y)$ , which can be proved by integrating both sides over a small square centered at the origin. When there is no ambiguity, we leave out the subindex  $N$  in matrices such as  $\mathbf{A}_N$ .

The resolvent  $\mathbf{G}_{\mathbf{A}}(\lambda)$  is not properly defined at the eigenvalues of  $\mathbf{A}$ , which are distributed over the complex plane. Therefore, one cannot apply the usual resolvent manipulations, as done for Hermitian matrices [25], to the non-Hermitian case. We can overcome this problem through an Hermitization method [8, 29, 30]. In this method one considers the resolvent  $\mathbf{G}_{\mathbf{B}}(\eta)$  of the  $2N \times 2N$  Hermitian block matrix  $\mathbf{B}_{2N}$

$$\mathbf{B}_{2N} = \begin{pmatrix} \mathbf{0}_N & \mathbf{A}_N - \lambda \\ \mathbf{A}_N^\dagger - \lambda^* & \mathbf{0}_N \end{pmatrix}, \quad (2)$$

where  $\mathbf{0}_N$  is an  $N \times N$  matrix filled with zeros and  $\eta \in \mathbb{R}$  is a regularizer which keeps  $\mathbf{G}_{\mathbf{B}}(\eta)$  properly defined on the whole complex plane. The resolvent  $\mathbf{G}_{\mathbf{A}}(\lambda)$  follows simply from the  $N \times N$  lower-left block of  $\lim_{\eta \rightarrow 0} \mathbf{G}_{\mathbf{B}}(\eta)$ , where  $\mathbf{G}_{\mathbf{B}}(\eta)$  are now computed using standard techniques for Hermitian matrices [31].

We associate a graph to the matrix  $\mathbf{A}_N$  by connecting all sites  $(i, j)$  with a nonzero  $A_{ij}$  element. If this graph has a local tree structure, we have the following exact expression for the spectrum  $\rho(\lambda) = -\lim_{N \rightarrow \infty, \eta \rightarrow 0} (\pi N)^{-1} \partial^* \sum_{i=1}^N [\mathbf{G}_i]_{21}$ , where the  $2 \times 2$  matrices  $\{\mathbf{G}_i\}$  fulfill the closed set of equations,

$$\mathbf{G}_i^{-1} = -\lambda(\eta) - \sum_{\ell \in \partial_i} \mathbf{A}_{i\ell} \mathbf{G}_\ell^{(i)} \mathbf{A}_{\ell i}, \quad (3)$$

$$(\mathbf{G}_i^{(j)})^{-1} = \mathbf{G}_i^{-1} + \mathbf{A}_{ij} \mathbf{G}_j^{(i)} \mathbf{A}_{ji}, \quad (4)$$

with  $\lambda(\eta) = \begin{pmatrix} i\eta & \lambda \\ \lambda^* & i\eta \end{pmatrix}$  and  $\mathbf{A}_{i\ell} = \begin{pmatrix} 0 & A_{i\ell} \\ A_{\ell i}^* & 0 \end{pmatrix}$ . From the point of view of random graphs, the matrix element  $A_{\ell i}$  is the weight of the directed edge from node  $\ell$  to  $i$ , while  $\partial_i$  contains the indices of the vertices belonging to the neighborhood of node  $i$ . We could derive Eqs. (3) and (4) from using recursively the Schur-complement formula [31] and we refer the reader to [33] for technical details.

Equations (3) and (4) have been derived for the first time by Rogers and Castillo [21] using an approach coming from the theory of spin glasses. For most random matrix ensembles, these equations have a highly intricate structure. To solve them analytically is unfeasible without further simplifications. The authors of [21] have confirmed exactness of Eqs. (3-4) by finding an excellent agreement with direct diagonalization methods for sparse matrices with a local tree structure. In contrast, here

we provide an analytical solution to these equations for a particular graph structure, which leads to an explicit formula for  $\rho(\lambda)$ .

In order to extract an interesting solution from Eqs. (3) and (4), capable to interpolate smoothly between a fully undirected and a fully directed graph, we now consider an ensemble of random sparse non-Hermitian matrices  $\mathbf{A}_N$  of dimension  $N \times N$  having the following constraints on its nondiagonal complex entries  $A_{ij}$

- $(A_{ij}, A_{ji}) \in \{(A_+, A_-), (A_-, A_+), (0, 0)\}$ ,
- $\sum_{j(\neq i)} \delta(A_{ij}, A_\pm) = \sum_{j(\neq i)} \delta(A_{ji}, A_\pm) = k$ ,

with  $k > 1$  and the diagonal elements  $A_{ii} = 0$ . This random ensemble can be represented by a graph of vertices  $[1..N]$  and directed segments connecting adjacent vertices  $(i, j)$  when  $A_{ij} \neq 0$ . In the corresponding graph every pair of adjacent vertices is connected by two edges in such a way that each vertex contains  $k$  incoming (outgoing) edges of weight  $A_+$  ( $A_-$ ) and  $k$  outgoing (incoming) edges of weight  $A_-$  ( $A_+$ ). Hence, the disorder in this ensemble is due to the topology of the underlying graph, while the elements  $A_{ij}$  are fixed to either of two values  $A_\pm$ . When the entries are real we have a polarized Bethe lattice, illustrated in Fig. 1. For  $A_- = A_+$  we recover the unoriented Bethe lattice [35] of degree  $2k$ , while for  $A_- = 0$  we obtain a fully oriented Bethe lattice of in- and out-degree equal to  $k$ .

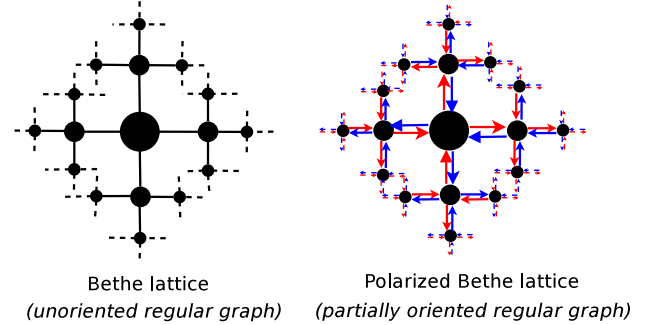


FIG. 1. (color online). An example of a Bethe lattice and a polarized Bethe lattice, both with degree  $k = 2$ . The matrix elements corresponding with the red (blue) lines are given by  $A_+$  ( $A_-$ ).

The resolvent equations (3) and (4) for this non-Hermitian ensemble become

$$\mathbf{G}^{-1} = -\lambda(\eta) - k(\mathbf{A}_- \mathbf{G}_+ \mathbf{A}_+ + \mathbf{A}_+ \mathbf{G}_- \mathbf{A}_-), \quad (5)$$

$$\mathbf{G}_\pm^{-1} = \mathbf{G}^{-1} + \mathbf{A}_\pm \mathbf{G}_\mp \mathbf{A}_\mp, \quad (6)$$

where  $\mathbf{A}_\pm = \begin{pmatrix} 0 & A_\pm \\ A_\mp^* & 0 \end{pmatrix}$ , and the spectrum follows from  $\rho(\lambda) = -\pi^{-1} \lim_{\eta \rightarrow 0} \partial^* [\mathbf{G}]_{21}$ . The simplified equations Eqs. (5) and (6) follow from Eqs. (3) and (4) after considering the transitive structure of the polarized

Bethe ensembles for  $N \rightarrow \infty$ . Equations (5) and (6) determine, for  $N \rightarrow \infty$ , the spectrum of a typical matrix drawn from our ensemble, since the resolvent equations constitute the typical local neighborhoods of our ensemble. We now determine the solution to the equations (5) and (6).

*The analytical expression for the spectrum* In the following we use polar coordinates and set  $A_{\pm} = p_{\pm} \exp(i\theta_{\pm})$ . Solving Eqs. (5) and (6) we find the analytical expressions for  $\mathbf{G}$  and  $\mathbf{G}_{\pm}$ , see supplemental material [34]. The spectrum  $\rho_0(\lambda)$  for real matrices  $\mathbf{A}_N$  (with  $\theta_+ = \theta_- = 0$ ) is given by

$$\rho_0(\lambda) = \frac{2kHp_+p_- \left[ \left( \frac{x}{S_+} \right)^2 - \left( \frac{y}{S_-} \right)^2 \right] + CW}{\pi \left[ \left( \frac{y}{S_-} \right)^2 + \left( \frac{x}{S_+} \right)^2 + C \right]^2 Q_+ Q_-} \quad (7)$$

for

$$x^2 Q_+^{-2} + y^2 Q_-^{-2} < H^{-1}, \quad (8)$$

and  $\rho_0(\lambda) = 0$  otherwise. In Eqs. (7) and (8) we have defined the constants  $H$ ,  $C$ ,  $W$ ,  $S_{\pm}$  and  $Q_{\pm}$ , which depend upon  $k$  and  $p_{\pm}$  as follows:

$$\begin{aligned} 2H &= k(p_+^2 + p_-^2) + \sqrt{k^2(p_+^2 - p_-^2)^2 + 4(k-1)^2(p_+p_-)^2}, \\ C &= k^2(k-1)^{-1}H^{-1} \left[ (p_+^2 + p_-^2)H - 2(p_+p_-)^2 \right], \\ W &= [H^2 + (2k-1)(p_+p_-)^2], \\ Q_{\pm} &= H \pm (2k-1)p_+p_-, \\ S_{\pm}^2 &= Q_{\pm}^2 [(H \mp p_+p_-)^2 - HC]^{-1}. \end{aligned}$$

Eqs. (7) and (8) are the main result of our work. The support (8) follows from a stability analysis of the trivial solution to Eqs. (5) and (6). Indeed, for large values of  $\lambda$  we find one stable trivial solution with  $\rho_0(\lambda) = 0$ . This solution is unstable at the support of  $\rho_0(\lambda)$ , see [34]. In Figs. 2 and 3 we compare direct diagonalization results with, respectively, the analytical expressions for the support (8) and the spectrum (7). We find a very good correspondence in both cases. Similar to Girko's law, the support forms an ellipse, but  $\rho_0(\lambda)$  is non-uniform.

Remarkably, for complex entries  $A_{\pm}$  ( $\theta_{\pm} \neq 0$ ) the spectrum  $\rho(\lambda)$  and its support follow simply from Eqs. (7) and (8) through a clockwise rotation by an angle  $\theta \equiv (\theta_+ + \theta_-)/2$  in the  $(x, y)$ -coordinate system, such that  $\rho(\lambda) = \rho_0(\lambda e^{-i\theta})$ . Results for  $\theta \neq 0$  are visualized in Fig. 2.

We discuss below a couple of interesting limiting cases:

1. *Fully directed Bethe lattice:* This limit is obtained by setting  $p_- = 0$  in Eqs. (7) and (8)

$$\rho_{\text{DB}}(\lambda) = \frac{k-1}{\pi} \left( \frac{k p_+}{|\lambda|^2 - k^2 p_+^2} \right)^2, \quad (9)$$

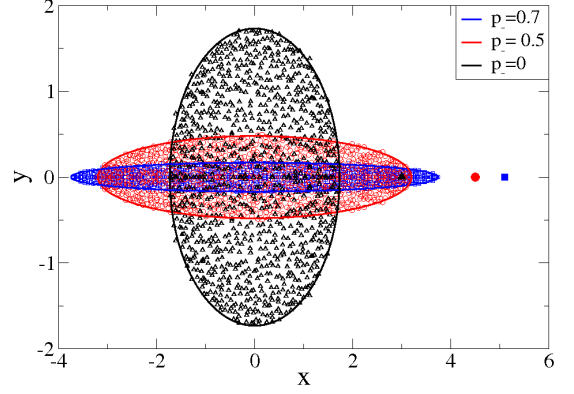


FIG. 2. (color online). Direct diagonalization results of matrices of size  $\mathcal{O}(1e+3)$  (symbols) compared with the support (8) (lines) for  $k=3$ ,  $p_+=1$  and different values of  $p_-$ .

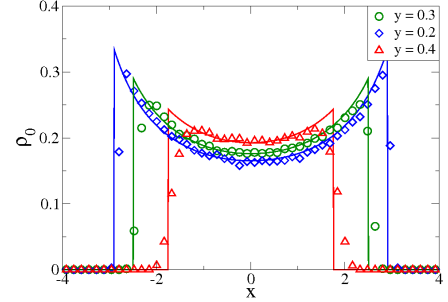


FIG. 3. (color online). Three cuts of  $\rho_0(\lambda)$ , Eq. (7), along the real direction are compared with direct diagonalization results of matrices of size  $\mathcal{O}(1e+3)$ , averaged over  $\mathcal{O}(1e+4)$  samples. The parameters are:  $k=3$ ,  $p_+=1$ ,  $p_-=0.5$ .

with the support  $|\lambda|^2 < kp_+^2$ . This formula has been conjectured before in [8, 36], but its rigorous proof remains an open problem [8].

2. *Undirected Bethe lattice:* By taking the limit  $p_+ \rightarrow p_- \equiv p$  in Eqs. (7) and (8), we obtain the Kesten-McKay law [12] for a graph with degree  $2k$

$$\rho_{\text{KM}}(\lambda) = \delta(y) \frac{k}{\pi} \frac{\sqrt{4p^2(2k-1) - x^2}}{4k^2p^2 - x^2}, \quad (10)$$

with support  $|x| < 2|p|\sqrt{2k-1}$ . For  $\theta \neq 0$  and  $p_+ \rightarrow p_-$ , the supports of the spectra in Fig. 2 reduce to straight lines along the major axes of the ellipses. The projected spectrum is then given by the Kesten-McKay measure. This concentration of all eigenvalues on the straight line  $y = x \tan(\theta)$  for a given  $\theta$  is a general property of matrices of the type  $\mathbf{A} + \exp(i\theta)\mathbf{A}^T$  ( $[\mathbf{A}]_{ij} \in \mathbb{R}$ ). Therefore, the eigenvalues can be brought to the real axis by a rotation. Non-Hermitian matrices with a real spectrum have attracted considerable attention as alternative theories to quantum mechanics [37].

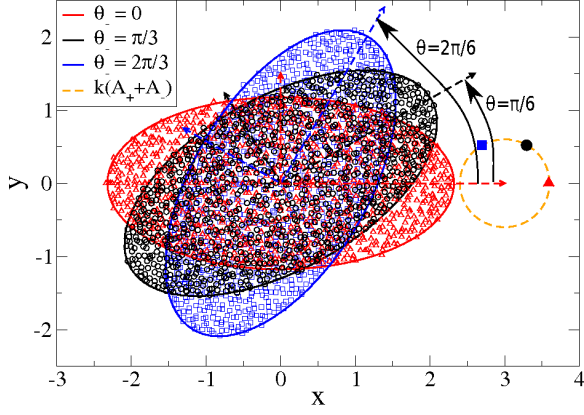


FIG. 4. (color online). Direct diagonalization results of matrices of size  $\mathcal{O}(1e+3)$  (symbols) compared with the rotated support of (8) (solid lines) for  $k=3$ ,  $p_+=1$ ,  $\theta_+=0$ ,  $p_-=0.2$  and different values of  $\theta_-$ . The orange circle denotes the eigenvalues isolated from the bulk, given by  $k(A_+ + A_-)$ . We notice the non-trivial rotation of the bulk spectrum of an angle  $\theta$ , different from the rotation of the isolated eigenvalue of an angle  $2\theta$  around  $k$ .

3. *Dense matrices:* By rescaling  $p_{\pm} \rightarrow p_{\pm}/\sqrt{k}$  in Eqs. (7) and (8), we obtain the highly connected limit when  $k \rightarrow \infty$

$$\rho(\lambda) = \frac{1}{\pi} \left( \frac{p_+^2 + p_-^2}{p_+^2 - p_-^2} \right)^2, \quad (11)$$

with support  $\frac{x^2}{(p_+ + p_-)^2} + \frac{y^2}{(p_+ - p_-)^2} < \frac{1}{p_+^2 + p_-^2}$ . When we set  $\mathbb{E}(A_{ij}^2) = 1$  and  $\mathbb{E}[A_{ij}A_{ji}] = \tau$ , with  $\mathbb{E}(\dots)$  denoting the ensemble average and  $-1 \leq \tau \leq 1$ , Eq. (11) reduces to  $\rho(\lambda) = [\pi(1 - \tau^2)]^{-1}$  with support  $x^2/(1 + \tau)^2 + y^2/(1 - \tau)^2 < 2$  [25]. Thus, Eqs. (7) and (8) form a non-trivial sparse realization of Girko's elliptic law. Indeed, for  $k \rightarrow \infty$  we have highly connected sparse ensemble leading to Girko's elliptic law. Sparse realizations of Wigner's semicircular law and Girko's circular law has recently been proven [38, 39].

*Biased diffusion on a regular graph* As an application we determine the convergence rate to the stationary state of a non-equilibrium transport process on a regular graph. Consider a set of random walkers moving along the edges of our partially-oriented regular graph model with  $\theta_{\pm} = 0$  and transition rates  $p_{\pm} > 0$ . The relative occupancies  $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_N)$ , with  $\pi_i$  the relative occupancy of the  $i$ -th site, fulfill the linear equation

$$\frac{d}{dt} \boldsymbol{\pi} = \mathbf{L}_N \boldsymbol{\pi}, \quad \mathbf{L}_N = \mathbf{A}_N - k(p_+ + p_-) \mathbf{1}_N. \quad (12)$$

Due to the Perron-Frobenius theorem, the Laplacian matrix  $\mathbf{L}_N$  has a unique eigenvector with positive entries and eigenvalue  $\lambda_0 = 0$ , which corresponds to the stationary solution of Eq. (12). All other eigenvalues have a negative real part. If we order them as  $\lambda_0 > \text{Re}(\lambda_1) >$

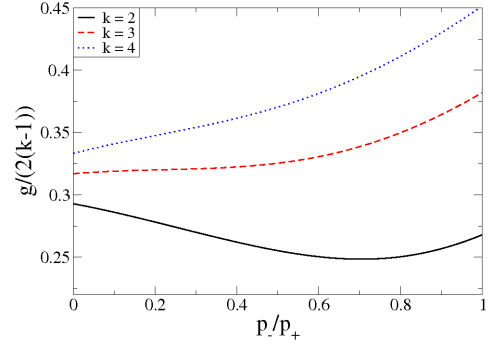


FIG. 5. (color online). Spectral gap as a function of the degree of symmetry  $\alpha = p_-/p_+$  for  $p_+ = 1$  and  $k$  given. When  $k \rightarrow \infty$ , the curves become straight lines.

$\dots > \text{Re}(\lambda_{N-1})$ , then we define the spectral gap as  $g = |\text{Re}(\lambda_1)|$ . This diffusion process converges exponentially to the steady state at a rate  $g$ .

We use Eq. (8) for the boundary to obtain the spectral gap for  $N \rightarrow \infty$

$$\frac{g}{p_+} = (1 + \alpha)k - \frac{[h^2(\alpha, k) + (2k - 1)\alpha]}{h(\alpha, k)}, \quad (13)$$

assuming that finite-size effects at the boundaries of  $\rho_0(\lambda)$  are negligible for large  $N$  [8]. The quantity  $\alpha = p_-/p_+$  is the degree of symmetry of the diffusion process and

$$2h^2(\alpha, k) = k(1 + \alpha^2) + \sqrt{k^2(1 - \alpha^2)^2 + 4(k - 1)^2\alpha^2}. \quad (14)$$

We plot  $g$  as a function of  $\alpha$  in Fig. 5. Remarkably, the speed of convergence is non-monotonic as a function of  $\alpha$  for  $k=2$ , indicating that for intermediate values of  $\alpha$  the diffusion process converges more slowly to the steady state. For  $k > 2$  the fully symmetric process has the fastest convergence. For dense random matrices such as the Ginibre ensemble the spectral gap is zero. Hence, we have determined analytically a physical property absent in previous studies.

*Conclusions* We have presented the analytical expression for the spectrum of a sparse non-Hermitian random matrix ensemble, which reduces to the Kesten-McKay law for Hermitian matrices and to the Girko's elliptic law in its highly connected limit. Previous studies for sparse non-Hermitian random matrix ensembles relied on numerical results. Using a specific distribution for the matrix elements we have found to our knowledge a first analytical expression for a non-Hermitian random matrix ensemble generalizing the Kesten-McKay measure. Such a result can stimulate research in two different directions. On the one hand, we have presented a random graph model which allows to address analytically how various processes on networks depend on the orientation of the edges. We have illustrated this through a study of the



convergence rate of a biased transport process. Examples that deserve further study include network synchronization [23], neural networks [24, 25] and non-Hermitian quantum mechanics [26–28]. On the other hand, since our result is derived through a heuristic approach, it poses a challenge for the development of new rigorous methods which deal with sparse non-Hermitian random matrices [8]. Further interesting research directions are: the development of exact results for eigenvector localization [40] and distribution of the largest eigenvalue [41] of sparse non-Hermitian matrices.

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# Supplemental material 1

## Derivation of the resolvent equations

I. Neri and F. L. Metz

### INTRODUCTION

Consider a large matrix  $\mathbf{A}_N$  of size  $N$  with matrix elements  $A_{ij} = [\mathbf{A}_N]_{ij} \in \mathbb{C}$ . In this document we present the essential steps in deriving the following set of resolvent equations

$$\mathbf{G}_i^{-1} = -\boldsymbol{\lambda}(\eta) - \sum_{\ell \in \partial i} \mathbf{A}_{i\ell} \mathbf{G}_\ell^{(i)} \mathbf{A}_{\ell i}, \quad (1)$$

$$(\mathbf{G}_i^{(j)})^{-1} = \mathbf{G}_i^{-1} + \mathbf{A}_{ij} \mathbf{G}_j^{(i)} \mathbf{A}_{ji}, \quad (2)$$

with  $\boldsymbol{\lambda}(\eta) = \begin{pmatrix} i\eta & \lambda \\ \lambda^* & i\eta \end{pmatrix}$  and  $\mathbf{A}_{i\ell} = \begin{pmatrix} 0 & A_{i\ell} \\ A_{\ell i}^* & 0 \end{pmatrix}$ . The set  $\partial_i$  is defined as  $\partial_i = \{j \in [1..N] | A_{ji} \neq 0 \vee A_{ij} \neq 0\}$ . For more details and the definition of other quantities we refer to the main paper.

The interest in the set of equations (1-2) is that it determines the spectrum of  $A_N$  through

$$\rho(\lambda) = - \lim_{N \rightarrow \infty, \eta \rightarrow 0} \frac{1}{\pi N} \partial^* \sum_{i=1}^N [\mathbf{G}_i]_{21}, \quad (3)$$

where  $\partial^* = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ . It is also possible to determine other quantities such as the diagonal correlator of eigenvectors [1], but we do not elaborate on this point in this paper.

The equations (1) and (2) have been derived for the first time in the recent work [2]. However, a rigorous derivation of these equations has not been presented yet. The conjecture is that they are exact when  $N \rightarrow \infty$  and when the underlying graph structure is locally tree-like. By the underlying graph we mean the connectivity graph one can form when connecting all points  $(i, j)$  which have  $A_{ij} \neq 0$ . The exactness of Eqs. (1) and (2) is strongly supported by a very good agreement between their numerical solution and direct diagonalization methods [2]. The concept of a local-tree like structure is properly defined in [3] and intuitively it means that most of the vertices have a finite neighborhood, where short loops are absent. This is true for randomly drawn graphs when they are very large [3]. Of course it is true that many real-world graphs are not constructed by

a random algorithm and contain an intricate loop structure. Nevertheless, it remains interesting to consider random systems as a first model for complex systems.

In this supplemental material we present an alternative derivation of the set of equations (1-2). The main difference between our method and the approach presented in [2] is that we avoid a mapping on a statistical mechanics problem. Therefore, the assumptions become more intuitive as we avoid the factorization of a complex valued function [2], which plays a role analogous to local marginals in spin models. Another advantage of our approach is that it stands much closer to resolvent methods used for rigorous spectral calculations on dense matrices [4].

## HERMITIZATION PROCEDURE

The spectral density of  $A_N$  is obtained from the resolvent  $\mathbf{G}_A(\lambda) \equiv (\lambda - \mathbf{A})^{-1}$  through equation  $\rho(\lambda) = \lim_{N \rightarrow \infty} (N\pi)^{-1} \partial^* \text{Tr} \mathbf{G}_A(\lambda)$ . The first step of the method consists in mapping the calculation of the spectrum of the non-Hermitian matrix  $\mathbf{A}_N$  on a resolvent calculation of the Hermitian matrix  $\mathbf{B}_{2N}$  [5], defined as

$$\mathbf{B}_{2N} = \begin{pmatrix} \mathbf{0}_N & \mathbf{A}_N - \lambda \\ \mathbf{A}_N^\dagger - \lambda^* & \mathbf{0}_N \end{pmatrix}, \quad (4)$$

where  $\mathbf{0}_N$  is a  $N \times N$  matrix filled with zeros and  $\eta \in \mathbb{R}$  is a regulator which keeps all quantities properly defined. In this framework, the spectrum follows from

$$\rho(\lambda) = -(N\pi)^{-1} \partial^* \lim_{\eta \rightarrow 0} \text{Tr} \left[ \begin{pmatrix} \mathbf{0}_N & \mathbf{1}_N \\ \mathbf{0}_N & \mathbf{0}_N \end{pmatrix} \mathbf{G}_B(\eta) \right]. \quad (5)$$

The Hermitization procedure can also be presented graphically, as can be seen in figure 1. It corresponds to the translation of an oriented graph to an unoriented bipartite

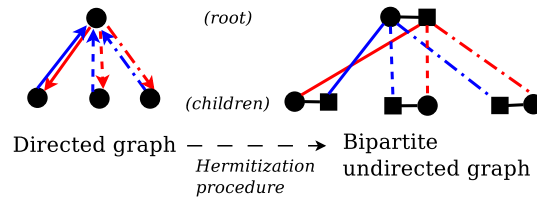


FIG. 1. (color online). Left: A non-Hermitian graph with oriented edges. Right: The undirected bipartite graph resulting from the application of the Hermitization procedure [5] to the subgraph on the left.

graph. Indeed, we can associate to the matrix  $\mathbf{A}_N$  a graph  $G = (V, E, W)$ , with the set of vertices  $V = [1, N]$  and the set of edges  $E = \{(i, j) | A_{ij} \neq 0\}$ . Each edge has a weight  $w_{ij} = A_{ij}$ , which is denoted by the mapping  $W : E \rightarrow \mathbb{C}$ . The edge is undirected when

$w_{ij} = w_{ji}^*$ , otherwise it is directed. Therefore, the translation  $\mathbf{A}_N \rightarrow \mathbf{B}_{2N}$  corresponds to a translation from an oriented graph to a bipartite unoriented graph.

## RECURSIVE APPLICATION OF THE SCHUR-COMPLEMENT FORMULA

From equation (??) we see that to calculate the spectrum, we need to have an expression for the diagonal elements of the resolvent  $\mathbf{G} = (\eta I_{2N} - \mathbf{B}_{2N})^{-1}$ . Since this involves the inverse of a matrix, the following formula is very useful:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}. \quad (6)$$

The above formula is referred to as the Schur-complement formula, and it is a common tool in the determination of spectra of dense matrices [4]. Recently the Schur-complement formula has also been applied to derive the Kesten-Mckay law [6]. Here we present a calculation similar to the one presented in [6].

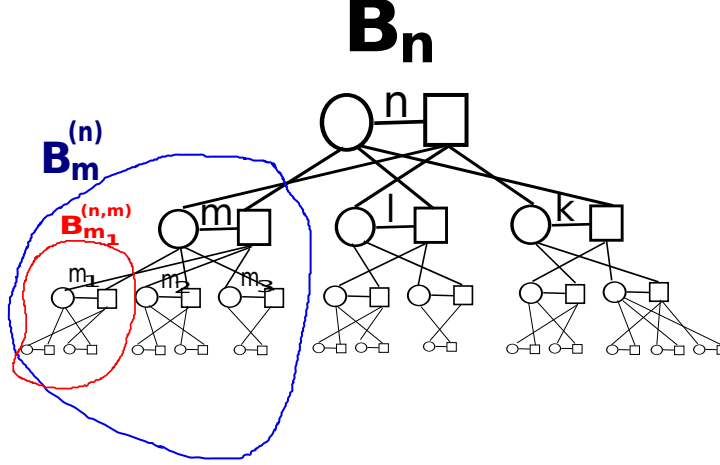


FIG. 2. A graphical representation of the recursive application of the Schur-complement formula on the bipartite matrix  $\mathbf{B}_{2N}$ . We present here the rooted graph around vertex  $n$ , corresponding to the matrix  $\mathbf{B}_n$ . We can close the Schur-complement recursion by setting  $\mathbf{B}_{m_i}^{(m,n)} = \mathbf{B}_{m_i}^{(m)}$ .

When relabeling the indices of a matrix we do not change its eigenvalues. We use this property and relabel all the vertices such that  $(i, i + N) \rightarrow (2i, 2i + 1)$ . It becomes useful to label each couple  $(2i, 2i + 1)$  by  $i$  and write  $\mathbf{B} - \eta I$  as a matrix consisting of  $2 \times 2$  blocks. In this matrix we have on the diagonal the  $2 \times 2$  matrix  $\boldsymbol{\lambda}(\eta) = \begin{pmatrix} i\eta & \lambda \\ \lambda^* & i\eta \end{pmatrix}$  while



the  $ij$ -th block is denoted by  $\mathbf{A}_{ij} = \begin{pmatrix} 0 & A_{ij} \\ A_{ji}^* & 0 \end{pmatrix}$ . We therefore find

$$\mathbf{B}'_{2N} = \begin{pmatrix} -\lambda & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1(N-1)} & \mathbf{A}_{1N} \\ \mathbf{A}_{21} & -\lambda & \cdots & \mathbf{A}_{2(N-1)} & \mathbf{A}_{2N} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{A}_{(N-1)1} & \mathbf{A}_{(N-1)2} & \cdots & -\lambda & \mathbf{A}_{(N-1)N} \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \cdots & \mathbf{A}_{N(N-1)} & -\lambda \end{pmatrix} \quad (7)$$

where the prime denotes the fact that we have relabeled the indices in  $\mathbf{B}_{2N}$ . In fact, the matrix  $\mathbf{B}'_{2N}$  is the relabeled matrix following from the above permutation operation on the matrix  $\eta I - \mathbf{B}_{2N}$ . Let us consider now the unoriented graph  $G' = (V', E', W')$ , with  $V' = [1..N]$ ,  $E' = \{(i, j) | A_{ij} \neq 0\}$  and  $w_{ij} = 1$  for all edges. We perform now a first depth search around a certain root vertex  $n$  in this unoriented graph and label the vertices accordingly. We define the rooted matrix  $\mathbf{B}^{(n)} = (\mathbf{B}', n)$ , as the matrix which we have created from  $\mathbf{B}'$  through a permutation according to the first-depth search around  $n$ . We have therefore for  $\mathbf{B}^{(n)}$ :

$$\mathbf{B}^{(n)} = \begin{pmatrix} -\lambda & \mathbf{A}_{nn_1} & \mathbf{0}_2 & \cdots & \mathbf{A}_{nn_2} & \mathbf{0}_2 & \cdots & \cdots & \mathbf{A}_{nn_{k_n}} & \mathbf{0}_2 & \cdots \\ \mathbf{A}_{n_1n} & & & & & & & & & & \\ \mathbf{0}_2 & \mathbf{B}_{n_1}^{(n)} & & & & & & & & & \\ \vdots & & & & & & & & & & \\ \mathbf{A}_{n_2n} & & & & & & & & & & \\ \mathbf{0}_2 & & \mathbf{B}_{n_2}^{(n)} & & & & & & & & \\ \vdots & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ \mathbf{A}_{n_{k_n}n} & & & & & & & & & & \\ \mathbf{0}_2 & & & & & & & & \mathbf{B}_{k_n}^{(n)} & & \\ \vdots & & & & & & & & & & \end{pmatrix}. \quad (8)$$

We have introduced the degree  $k_n$ , which denotes the number of matrix elements  $A_{ni}$ ,  $i(\neq n) \in V$ , which are different than zero. The matrices  $\mathbf{B}_{n_i}^{(n)}$  correspond to a subgraph of  $G'$  associated to one of the children nodes (or branches) of  $n$ . This picture is visualized in figure 1.

We apply now the Schur-complement formula, to find:

$$\left( \begin{bmatrix} \left( \mathbf{B}^{(n)} \right)^{-1} \\ \left( \mathbf{B}^{(n)} \right)^{-1} \end{bmatrix}_{21}^{11} \begin{bmatrix} \left( \mathbf{B}^{(n)} \right)^{-1} \\ \left( \mathbf{B}^{(n)} \right)^{-1} \end{bmatrix}_{22}^{12} \right) = \frac{1}{-\lambda - \sum_{j=1}^{k_n} (\mathbf{A}_{nn_j} \mathbf{0}_2 \cdots \mathbf{0}_2) \left( \mathbf{B}_{n_j}^{(n)} \right)^{-1} (\mathbf{A}_{nn_j} \mathbf{0}_2 \cdots \mathbf{0}_2)^\dagger} \quad (9)$$

We see that to determine the right hand side of equation (9), it is necessary to have an expression in the upper  $2 \times 2$  dimensional block of the inverse of the matrix  $\mathbf{B}_{n_1}^{(n)}$ , i. e., we need to know the elements  $\left[\left(\mathbf{B}_{n_1}^{(n)}\right)^{-1}\right]_{11}$ ,  $\left[\left(\mathbf{B}_{n_1}^{(n)}\right)^{-1}\right]_{12}$ ,  $\left[\left(\mathbf{B}_{n_1}^{(n)}\right)^{-1}\right]_{21}$  and  $\left[\left(\mathbf{B}_{n_1}^{(n)}\right)^{-1}\right]_{22}$ . These elements can be determined using again the Schur-complement formula (hence the recursion). Since  $\mathbf{B}_{n_1}^{(n)}$  is the matrix corresponding to one of the subgraphs of  $G'$ , we have (setting  $n_1 = m$ )

$$\mathbf{B}_m^{(n)} = \begin{pmatrix} -\lambda & \mathbf{A}_{mm_1} & \mathbf{0}_2 & \cdots & \mathbf{A}_{jj_2} & 0 & \cdots & \cdots & \mathbf{A}_{mm_{k_m}} & 0 & \cdots \\ \mathbf{A}_{mm_1}^\dagger & & & & & & & & & & \\ 0 & \mathbf{B}_{m_1}^{(n,m)} & & & & & & & & & \\ \vdots & & & & & & & & & & \\ \mathbf{A}_{mm_2}^\dagger & & & & & & & & & & \\ 0 & & \mathbf{B}_{m_2}^{(n,m)} & & & & & & & & \\ \vdots & & & & & & & & & & \\ \vdots & & & & & & & & & & \\ \mathbf{A}_{mm_{k_m-1}}^\dagger & & & & & & & & & & \\ 0 & & & & & & & & \mathbf{B}_{k_m-1}^{(n,m)} & & \\ \vdots & & & & & & & & & & \end{pmatrix} \quad (10)$$

After applying again the Schur-complement formula, we can close the resultant set of equations using  $\mathbf{B}_m^{(o)} = \mathbf{B}_m^{(n,o)}$ , for all  $n \in \partial_o$  and  $m \in \partial_o$ , with  $\partial_o$ . Using this approximation we find the closed set of equations (1-2). Indeed, setting

$$\mathbf{G}_n = \begin{pmatrix} \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{11} & \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{12} \\ \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{21} & \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{22} \end{pmatrix} \quad (11)$$

and

$$\mathbf{G}_m^{(n)} = \begin{pmatrix} \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{11} & \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{12} \\ \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{21} & \left[\left(\mathbf{B}_m^{(n)}\right)^{-1}\right]_{22} \end{pmatrix} \quad (12)$$

we recover indeed the equations (1-2).

We remark that the condition  $\mathbf{B}_m^{(o)} = \mathbf{B}_m^{(n,o)}$  is indeed exact on a tree. Therefore, our approximation has some clear intuition.

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# Supplemental material 2

## Solving the resolvent equations

I. Neri and F. L. Metz

### INTRODUCTION

In this supplemental material we present how the polarized Bethe lattice leads to a non-trivial solution to the resolvent equations. We present the exact analytical solution to the resolvent equations in this case.

### RESOLVENT EQUATIONS: GENERAL CASE

Let us consider a sparse non-Hermitian matrix  $\mathbf{A}$ , with  $A_{ij} = [\mathbf{A}]_{ij}$  and  $i, j = 1 \dots N$ . We define the set  $\partial_i = \{j | A_{ij} \neq 0 \vee A_{ji} \neq 0\}$ . The resolvent equations in the  $2 \times 2$  matrices  $\mathbf{G}_i$  and  $\mathbf{G}_j^{(i)}$  are given by

$$\mathbf{G}_i^{-1} = -\boldsymbol{\lambda}(\eta) - \sum_{\ell \in \partial i} \mathbf{A}_{i\ell} \mathbf{G}_\ell^{(i)} \mathbf{A}_{\ell i}, \quad (1)$$

$$(\mathbf{G}_i^{(j)})^{-1} = \mathbf{G}_i^{-1} + \mathbf{A}_{ij} \mathbf{G}_j^{(i)} \mathbf{A}_{ji} \quad (2)$$

with  $\boldsymbol{\lambda}(\eta) = \begin{pmatrix} i\eta & \lambda \\ \lambda^* & i\eta \end{pmatrix}$  and  $\mathbf{A}_{i\ell} = \begin{pmatrix} 0 & A_{i\ell} \\ A_{\ell i}^* & 0 \end{pmatrix}$ . These equations present an algorithm which allows us to determine, among other spectral quantities, the spectrum of the graph through

$$\rho(\lambda) = - \lim_{N \rightarrow \infty, \eta \rightarrow 0} \frac{1}{N\pi} \partial^* \sum_{i=1}^N [\mathbf{G}_i]_{21} \quad (3)$$

where  $\partial^* = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ . The equations (1) and (2) are conjectured to be exact when  $N \rightarrow \infty$  and when the graph is locally tree like. Its use as an efficient and accurate algorithm has been demonstrated in [1]. Here we present an analytical solution to Eqs. (1-2) for a partially-oriented regular graph. Our formula generalizes Girko's elliptic law to the sparse case as well as the Kesten-Mckay law to the non-Hermitian case.

# RESOLVENT EQUATIONS FOR PARTIALLY-ORIENTED REGULAR GRAPHS <sup>2</sup>

The resolvent equations for the partially-oriented regular graph, see figure 2 in the main paper, are given by

$$\mathbf{G}^{-1} = -\lambda(\eta) - k(\mathbf{A}_- \mathbf{G}_+ \mathbf{A}_+ + \mathbf{A}_+ \mathbf{G}_- \mathbf{A}_-), \quad (4)$$

$$\mathbf{G}_{\pm}^{-1} = \mathbf{G}^{-1} + \mathbf{A}_{\pm} \mathbf{G}_{\mp} \mathbf{A}_{\mp}. \quad (5)$$

These are equations (6-7) in the main paper. The set of equations (4-5) has only one stable solution. This has been proven for the Hermitian case [2], and we conjecture, for now, this to be also true in the non-Hermitian case. Something which is supported by numerically solving the resolvent equations. We find for large values of  $|\lambda|$  a trivial solution  $(\mathbf{G}_{\pm}^t, \mathbf{G}^t)$  (with  $\rho(\lambda) = 0$ ). At small values of  $|\lambda|$ , the trivial solution becomes unstable in favor of a non-trivial solution  $(\mathbf{G}_{\pm}^n, \mathbf{G}^n)$  (with  $\rho(\lambda) > 0$ ). Below we describe the trivial solution  $(\mathbf{G}_{\pm}^t, \mathbf{G}^t)$  and the non-trivial solution  $(\mathbf{G}_{\pm}^n, \mathbf{G}^n)$ . We also determine the line in the complex plane where the trivial solution becomes unstable. This line corresponds with the boundary of the support of the spectrum.

Let us first define some constants which we will use throughout:

$$\begin{aligned} U &\equiv \frac{A_+ A_-^* + A_+^* A_-}{2}, \\ V &\equiv \frac{A_+^* A_+ + A_-^* A_-}{2}, \\ W &\equiv \frac{A_- A_-^* - A_+ A_+^*}{2i}, \\ Z &\equiv \frac{A_+ A_- - A_+^* A_-^*}{2i}. \end{aligned}$$

We remind that  $A_+$  and  $A_-$  are the non-zero matrix elements or equivalently  $(A_{\pm}, A_{\mp})$  are the weights of the edges in the graph. Every vertex is incident to  $k$  edges of weight  $(A_+, A_-)$  and  $k$  edges of weight  $(A_-, A_+)$ . It is convenient to use polar coordinates and write down  $A_+ = p_+ \exp(i\theta_+)$  and  $A_- = p_- \exp(i\theta_-)$ .

At last we point out that it will be useful to parametrize the resolvent matrices as follow:

$$\mathbf{G}_{\pm} = \begin{pmatrix} a \pm b & c \\ d & a \mp b \end{pmatrix}, \quad (6)$$

$$\mathbf{G} = \begin{pmatrix} a' + b' & c' \\ d' & a' - b' \end{pmatrix}. \quad (7)$$



with  $(a, b, c, d) \in \mathbb{C}^4$ .

### THE TRIVIAL SOLUTION: $\rho(\lambda) = 0$

The trivial solution follows from substitution of equations (6-7) in (4-5) and setting  $a = b = 0$ . The trivial solution is therefore given by

$$\mathbf{G}_{\pm}^t = \begin{pmatrix} 0 & c_0 \\ d_0 & 0 \end{pmatrix}, \quad (8)$$

$$\mathbf{G}^t = \begin{pmatrix} 0 & c'_0 \\ d'_0 & 0 \end{pmatrix}, \quad (9)$$

with

$$c_0 = \frac{-\lambda^* + \sqrt{(\lambda^*)^2 - 4(2k-1)(U-iZ)}}{2(2k-1)(U-iZ)}, \quad (10)$$

$$d_0 = \frac{-\lambda + \sqrt{\lambda^2 - 4(2k-1)(U+iZ)}}{2(2k-1)(U+iZ)}, \quad (11)$$

and

$$c'_0 = -\frac{2k-1}{(k-1)\lambda^* + \sqrt{(\lambda^*)^2 - 4(2k-1)(U-iZ)}},$$

$$d'_0 = -\frac{2k-1}{(k-1)\lambda + \sqrt{\lambda^2 - 4(2k-1)(U+iZ)}}.$$

The spectrum follows from  $\rho(\lambda) = -\pi^{-1} \frac{\partial}{\partial \lambda^*} d'_0 = 0$ .

**THE NON-TRIVIAL SOLUTION:  $\rho(\lambda) > 0$**

The non-trivial solution follows from substitution of equations (6-7) in (4-5) and setting  $a \neq 0$  and  $b \neq 0$ :

$$\begin{aligned} a^2 &= \left( c d - \frac{1}{H} \right) \left( \frac{(H - V)^2}{W^2 + (H - V)^2} \right), \\ b^2 &= - \left( c d - \frac{1}{H} \right) \left( \frac{W^2}{W^2 + (H - V)^2} \right), \\ c &= \frac{(2k - 1)U \lambda^* + (Z(2k - 1) - H)\lambda}{H^2 - (2k - 1)^2(U^2 + Z^2)}, \\ d &= \frac{-H\lambda^* + (-iZ(2k - 1) + (2k - 1)U)\lambda}{H^2 - (2k - 1)^2(U^2 + Z^2)}, \end{aligned}$$

and for the resolvent:

$$\begin{aligned} (a')^2 &= \tilde{M}^{-2} 4k^2 \left[ -(-x(-(2k - 1)U + H) + Z(2k - 1)y)^2 \right. \\ &\quad \left. - (y((2k - 1)U + H) - Z(2k - 1)x)^2 + \frac{H^2 - (2k - 1)^2(U^2 + Z^2)}{H} \right] \left[ \frac{(V(H - V) - W^2)^2}{W^2 + (H - V)^2} \right], \end{aligned}$$

$$b' = 0,$$

$$\begin{aligned} c' &= \tilde{M}^{-1} \left[ y(2kZH) + x(-H^2 - (2k - 1)(U^2 + Z^2) + 2kUH) \right. \\ &\quad \left. - i(y(H^2 + (2k - 1)(U^2 + Z^2) + 2kUH) + x(-2kHZ)) \right], \end{aligned}$$

$$\begin{aligned} d' &= \tilde{M}^{-1} \left[ y(2kZH) + x(-H^2 - (2k - 1)(U^2 + Z^2) + 2kUH) \right. \\ &\quad \left. + i(y(H^2 + (2k - 1)(U^2 + Z^2) + 2kUH) + x(-2kHZ)) \right]. \end{aligned}$$

The denominator is given by

$$\tilde{M} = \frac{M}{H^2 - (2k - 1)^2(U^2 + Z^2)}, \quad (12)$$

with

$$\begin{aligned}
M = 4k^2 & \left[ -(-x(-(2k-1)U+H) + Z(2k-1)y)^2 \right. \\
& \left. - (y((2k-1)U+H) - Z(2k-1)x)^2 + \frac{H^2 - (2k-1)^2(U^2 + Z^2)}{H} \right] \left[ \frac{(V(H-V) - W^2)^2}{W^2 + (H-V)^2} \right] \\
& + [y(2kZH) + x(-H^2 - (2k-1)(U^2 + Z^2) + 2kUH)]^2 \\
& + [y(H^2 + (2k-1)(U^2 + Z^2) + 2kUH) + x(-2kHZ)]^2. \tag{13}
\end{aligned}$$

We remind that  $\lambda = x + iy$ . The spectrum of the non-trivial solution follows then from  $\rho(\lambda) = -\pi^{-1} \frac{\partial}{\partial \lambda^*} d'$

$$\rho(\lambda) = \rho_0(\lambda e^{-i\theta}), \tag{14}$$

with  $\theta = \theta_+ + \theta_-$  and

$$\rho_0(\lambda) = \frac{2kHp_+p_- \left[ \left( \frac{x}{s_+} \right)^2 - \left( \frac{y}{s_-} \right)^2 \right] + CW}{\pi \left[ \left( \frac{y}{s_-} \right)^2 + \left( \frac{x}{s_+} \right)^2 + C \right]^2 Q_+ Q_-},$$

with constants

$$\begin{aligned}
2H &= k(p_+^2 + p_-^2) + \sqrt{k^2(p_+^2 - p_-^2)^2 + 4(k-1)^2(p_+p_-)^2}, \\
C &= k^2(k-1)^{-1}H^{-1}[(p_+^2 + p_-^2)H - 2(p_+p_-)^2], \\
W &= [H^2 + (2k-1)(p_+p_-)^2], \\
Q_{\pm} &= H \pm (2k-1)p_+p_-, \\
S_{\pm}^2 &= Q_{\pm}^2 [(H \mp p_+p_-)^2 - HC]^{-1}.
\end{aligned}$$

## STABILITY ANALYSIS: DERIVATION OF THE SUPPORT

The support is found through a stability analysis around the trivial solution (8-9). We make therefore an expansion around the trivial solution of the kind

$$a = a_1, \tag{15}$$

$$b = b_1, \tag{16}$$

$$c = c_0 + c_1, \tag{17}$$

$$d = d_0 + d_1, \tag{18}$$

with  $|a_1| \ll 1$ ,  $|b_1| \ll 1$ ,  $|c_1| \ll 1$  and  $|d_1| \ll 1$ . Substitution in equations (4-5), this<sup>6</sup> linear stability analysis leads to the following equation:

$$|(c_0^2 + d_0^2)| = H^{-1} \quad (19)$$

The expression in (??) is in fact an ellipse of the kind:

$$x^2 Q_+^{-2} + y^2 Q_-^{-2} < H^{-1}, \quad (20)$$

for  $\theta = 0$ , while for  $\theta > 0$  the ellipse rotates around the origin by an angle  $\theta$ . This is the formula presented in our main paper. One can see that Eq. (19) represents an ellipse rather quickly by considering that the conformal map  $\zeta = \lambda \pm \sqrt{\lambda^2 - d}$  maps a circle in the  $\zeta$ -space on an ellipse in the  $\lambda$ -space. Considering the formula for  $c_0$  and  $d_0$ , Eqs. (10-11), we notice indeed that (19) represents an ellipse.

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